Stat 155 Lecture 23 Notes

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1 Impossibility Theorems and Properties of Voting Systems

1.1 The Gibbard-Satterthwaite theorem

Last time we introduced Arrow's[1] Impossibility theorem.

Theorem 1.1 (Arrow's Impossibility theorem)**.** *For $|\Gamma| \geq 3$, any ranking rule R that satisfies both IIA and unanimity is a dictatorship.*

Here is another impossibility theorem.

Definition 1.1. A *voting rule* f is a function that takes the voters' preference profile π to the winner in Γ.

Definition 1.2. A voting rule f is *onto* the set Γ of candidates if, for all candidates $A \in \Gamma$, there is a preference profile π such that $f(\pi) = A$.

If f is not onto Γ, some candidate is excluded from winning.

Theorem 1.2 (Gibbard-Satterthwaite)**.** *For $|\Gamma| \geq 3$, any voting rule f that is onto Γ and is not strategically vulnerable is a dictatorship.*

Proof. The proof is by contradiction; we use f to construct a ranking rule that violates Arrow's theorem. Suppose f is onto Γ, not strategically vulnerable, and not a dictatorship. Define $\triangleright = R(\pi)$ via

$$\begin{cases} A \triangleright B & f(\pi^{\{A,B\}}) = A, \\ B \triangleright A & f(\pi^{\{A,B\}}) = B, \end{cases}$$

where π^S maintains the order of candidates in S but moves them above all other candidates in all voters' preferences.

If f is onto Γ and not strategically vulnerable, then for all $S \subseteq \Gamma$, $f(\pi^S) \in S$, so $\triangleright$ is complete; otherwise, in the path from a $\pi' \in f^{-1}(S)$ to π^S, some voter switch would

[1] Kenneth Arrow was a professor of Operations Research and Economics at Stanford. He won the Nobel Prize in Economics in 1972 and is considered the founder of modern social choice theory.

demonstrate a strategic vulnerability. Also, $\triangleright$ is transitive; the same argument shows that $f(\pi^{\{A,B,C\}}) = A$ implies $A \triangleright B$ and $A \triangleright C$, so cycles are impossible.

So R satisfies unanimity because $A \succ_i B$ implies that $\pi^{\{A,B\}} = (\pi^{\{A,B\}})^{\{A\}}$, so $A \triangleright B$. By a similar argument, R satisfies IIA. So by Arrow's impossibility theorem, R is a dictatorship. But because f is not a dictatorship, neither is R. So we have a contradiction. □

1.2 Properties of voting systems

Here are some more properties of voting systems. Are these desirable? Are they realistic?

Definition 1.3. A voting system is *symmetric* if permuting voters does not affect the outcome.

Definition 1.4. A voting system is *monotonic* if changing one voter's preferences by promoting candidate A without changing any other preferences should not change the outcome from A winning to A not winning.

Definition 1.5. The *Condorcet winner criterion* is that if a candidate is majority-preferred in pairwise comparisons with any other candidate, then that candidate wins.

Definition 1.6. The *Condorcet loser criterion* is that if a candidate is preferred by a minority of voters in pairwise comparisons with all other candidates, then that candidate should not win.

Definition 1.7. The *Smith criterion* is that the winner always comes from the *Smith set*, the smallest nonempty set of candidates that are majority-preferred in pairwise comparisons with any candidate outside the set.

Definition 1.8. A voting system is *reversal symmetric* if when candidate A wins for some voter preference profile, candidate A does not win when the preferences of all voters are reversed.

Definition 1.9. *Cancellation of ranking cycles* is when if a set of $|\Gamma|$ voters have preferences that are cyclic shifts of each other (e.g. $A \succ_1 B \succ_1 C$, $B \succ_2 C \succ_2 A$, and $C \succ_3 A \succ_3 B$), then removing these voters does not affect the outcome.

Definition 1.10. *Cancellation of opposing rankings* is when if two voters have reverse preferences, then removing these voters does not affect the outcome.

Definition 1.11. *Participation* is when if candidate A wins for some voter peferrence profile, then adding a voter with $A \succ B$ does not change the winner from A to B.

Example 1.1. Which of these properties does instant runoff voting have? Recall that in instant runoff voting, we eliminate the candidate that is top-ranked by the fewest voters, remove that candidate from everyone's rankings and repeat.

- Instant runoff voting satisfies symmetry because permuting the voters does not affect the outcome.
- Instant runoff voting does not satisfy monotonicity, however; our example from the last two lectures of strategic voting is a counterexample to monotonicity.
- Instant runoff voting does not satisfy the Condorcet winner criterion. Here is an example where B is preferred over any candidate, but A wins.

	1st	2nd	3rd
30%	A	B	C
45%	C	B	A
25%	B	A	C

- Instant runoff voting satisfies the Condorcet loser criterion. If the Condorcet loser makes it to the last round, they will lose the pairwise vote in that round; so they cannot win.
- Instant runoff voting does not satisfy the Smith criterion. In the above example, the Smith set is $\{B\}$, but A wins instead of B.
- Instant runoff voting is not reversal symmetric. In the following example, reversing the preferences still makes candidate A the winner.

	1st	2nd	3rd
30%	A	B	C
45%	C	B	A
25%	B	A	C

	1st	2nd	3rd
30%	C	B	A
45%	A	B	C
25%	C	A	B

1.3 Positional voting rules

Definition 1.12. A *positional voting rule* is defined as follows. Let $a_1 \geq a_2 \geq \cdots \geq a_N$. For each candidate, assign a_i points for each voter that assigns that candidate rank i. The candidate with the largest total wins.

Example 1.2. Borda[2] count is the positional voting rule with a_i given by $N, N-1, \ldots, 1$.

Example 1.3. Plurality is the positional voting rule with a_i given by $1, 0, \ldots, 0$.

Example 1.4. Approval voting is the rule with a_i given by $1, 1, \ldots, 1, 0, \ldots, 0$.

Positional voting rules satisfy symmetry, monotonicity and cancellation of ranking cycles. However, they do not necessarily satisfy the Condorcet winner criterion.

[2] Jean-Charles de Borda was an 18th century French naval commander, scientist, and inventor. He created ballistics, mapping and surveying instruments, pumps, and metric trigonometric tables.